Lattice QCD — Introduction and Results

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QCD – Gauge theory of the strong interaction

- Lagrangian: formulated in terms of quarks and gluons

\[ \mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} + \sum_f \overline{\psi}_f (i \gamma^\mu D_\mu - m_f) \psi_f, \quad f = u, d, s, c, b, t \]

\[ D_\mu = \partial_\mu - ig(\frac{1}{2} \lambda^a) A_\mu^a \]

- Parameters:
  - gauge coupling: \( g, \quad \alpha_s = g^2 / 4\pi \)
  - quark masses: \( m_u, m_d, m_s, \ldots \)

- Amazingly simple structure
Properties of QCD

Asymptotic freedom: $g(\mu)$

Confinement

[Eidelman et al., PDG 2004]

Nobel Prize in Physics 2004

“...for the discovery of asymptotic freedom in the theory of the strong interaction”

(c.f. D. Politzer: *The Dilemma of Attribution* — Nobel lecture)
\[ \mu \simeq 100 \text{GeV}: \text{weakly coupled quarks and gluons:} \]

\[ e^+ e^- \rightarrow \text{jets} \]

\[ \alpha_s(M_Z) = 0.121 \pm 0.007 \]

\[ \mu \simeq 855 \text{MeV}: \text{bound states of quarks and gluons:} \]

\[ \pi, K, \ldots, \rho, K^*, \ldots, P, \Sigma, \ldots, \Delta, \Sigma^*, \ldots, G(?) \]

\[ e p \rightarrow e p \gamma \]

Perturbation theory in \( \alpha_s \) not applicable!
- Connecting the low-energy to the high-energy regime of the strong interaction requires **non-perturbative** treatment

**Lattice QCD** [Wilson 1974]

→ Formulation of QCD on **discretised** space-time

→ Determination of observables using **numerical simulations**
Outline:

**Lattice QCD — The Method**

1. Lattice actions for QCD
2. Path integral & observables
3. Algorithms & machines

**Lattice QCD — Recent Results** *(selection!)*

4. Hadron spectrum
5. Strong coupling constant
6. Flavour physics
Lattice QCD — The Method
1. Lattice actions for QCD

Minkowski space-time, continuum $\rightarrow$ Euclidean space-time, discretised

Lattice spacing $a, \quad a^{-1} \sim \Lambda_{\text{UV}}, \quad x_\mu = n_\mu a$

Finite volume $L^3 \cdot T, \quad N_s = L/a, \quad N_t = T/a$

(anti)quarks: $\psi(x), \bar{\psi}(x)$

gluons: $U_\mu(x) = e^{aA_\mu(x)} \in \text{SU}(3)$

field tensor: $P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x)$

“plaquettes”

lattice sites

links
In lattice QCD the (non-Abelian) gauge field is represented by an SU(3) matrix:

\[ U_\mu(x) \in SU(3), \quad \text{(link variable)} \]

Gauge transformation:

\[ U_\mu(x) \rightarrow g(x)U_\mu(x)g(x + a\hat{\mu})^{-1}, \quad g(x), g(x + a\hat{\mu}) \in SU(3) \]

Let \( A_{\mu}^{\text{cont}}(x) \) be a given gauge potential in the continuum:

\[ U_\mu(x) = e^{aA_{\mu}^{\text{cont}}(x)}, \quad A_{\mu}^{\text{cont}}(x) = \lim_{a \rightarrow 0} \frac{1}{a}(U_\mu(x) - 1) \]

Formulate expressions for the QCD action in terms of link variables and fermionic fields

**Lattice action:** \( S[U, \bar{\psi}, \psi] = S_G[U] + S_F[U, \bar{\psi}, \psi] \)
Wilson “plaquette” action for Yang-Mills theory

\[ S_G[U] = \beta \sum_x \sum_{\mu < \nu} \left( 1 - \frac{1}{3} \text{Re} \, \text{Tr} \, P_{\mu\nu}(x) \right), \quad \beta = \frac{6}{g_0^2}, \quad (\text{gauge invariant}) \]

\[ P_{\mu\nu}(x) = U_\mu(x)U_\nu(x + a\hat{\mu})U_\mu^\dagger(x + a\hat{\nu})U_\nu^\dagger(x) \]

For small lattice spacings:

\[ S_G[U] \rightarrow -\frac{1}{2g_0^2} \int d^4x \text{Tr} \left[ F_{\mu\nu}(x)F_{\mu\nu}(x) \right] + O(a) \]

Proof: insert \( U_\mu(x) = e^{aA_\mu(x)} \) into \( P_{\mu\nu} \) and Taylor-expand in \( a \).

N.B. Discretisation not unique!
**Fermionic part:**

- Discretised version of the covariant derivative:
  \[
  \nabla_\mu \psi(x) \equiv \frac{1}{a} \left( U_\mu(x) \psi(x + a\hat{\mu}) - \psi(x) \right)
  \]
  \[
  \nabla^*_\mu \psi(x) \equiv \frac{1}{a} \left( \psi(x) - U_\mu^+(x - a\hat{\mu}) \psi(x - a\hat{\mu}) \right)
  \]

- "Naive" discretisation of fermionic part \( S_F \):
  \[
  D_{\text{naive}} + m_f = \frac{1}{2} \gamma_\mu \left( \nabla_\mu + \nabla^*_\mu \right) + m_f
  \]
  \[
  \tilde{D}_{\text{naive}}(p) = i\gamma_\mu \frac{1}{a} \sin(ap_\mu) = i\gamma_\mu p_\mu + O(a^2) \quad \text{(free theory)}
  \]
  \[
  \rightarrow \tilde{D}_{\text{naive}}(p) \text{ vanishes for } p_\mu = 0, \pi/a
  \]
  \[
  \rightarrow \text{produces } 2^4 = 16 \text{ poles in fermion propagator of flavour } f
  \]
  \[
  \rightarrow 16\text{-fold degeneracy of fermion spectrum: } \textbf{Fermion doubling problem}
  \]
Fermionic discretisations:


b. Staggered (Kogut-Susskind) fermions [Kogut+Susskind 1975]

c. Overlap/Domain Wall fermions [Kaplan '92, Furman+Shamir '96, Neuberger '98]

d. “Perfect”/Fixed point actions [Hasenfratz+Niedermaier '93/'98]

Wilson fermions

- Add a term to $D_{\text{naive}}$ which formally vanishes as $a \to 0$:

$$D_w + m_f = \frac{1}{2} \gamma_\mu (\nabla_\mu + \nabla^*_\mu) + ar \nabla^*_\mu \nabla_\mu + m_f$$

$$\tilde{D}_w(p) = i\gamma_\mu \frac{1}{a} \sin(ap_\mu) + \frac{2r}{a} \sin^2 \left( \frac{ap_\mu}{2} \right) \quad \text{(free theory)}$$

$\Rightarrow$ Mass of doubler states receives contribution $\propto r/a$: pushed to cutoff scale

$\Rightarrow$ Complete lifting of degeneracy

$\Rightarrow$ Explicit breaking of chiral symmetry:

- Even for $m_f = 0$ the action is no longer invariant under

$$\psi(x) \to e^{i\alpha \gamma_5} \psi(x), \quad \bar{\psi}(x) \to \bar{\psi}(x)e^{i\alpha \gamma_5}$$

Mostly acceptable, but makes things more complicated
Staggered (Kogut-Susskind) fermions

- Reduce d.o.f. by distributing single spinor components over corners of hypercube

\[ 16 \rightarrow 4 \Rightarrow 4 \text{ “tastes” per physical flavour} \]

- Flavour symmetry broken: gluons mix “tastes”

\[ \text{Remnant of chiral symmetry: global } U(1) \otimes U(1) \]
Chiral Symmetry on the Lattice

- Lattice regularisation: incompatible with chiral symmetry?
  
  Either: fermion doubling problem
  Or : explicit chiral symmetry breaking:  \( \{\gamma_5, D\} \neq 0 \)

  [Nielsen+Ninomiya 1979]

- Chiral symmetry at non-zero lattice spacing realised if


  \( \gamma_5 D + D\gamma_5 = aD\gamma_5 D \)

- Explicit construction: Neuberger-Dirac operator

  \[ D_N = \frac{1}{a} \left\{ 1 - \frac{A}{\sqrt{A^\dagger A}} \right\}, \quad A = 1 - aD_w \]

  \( D_w : \) massless Wilson-Dirac operator

  [Neuberger 1998]
\[ S_F[U, \bar{\psi}, \psi] = a^4 \sum_x \bar{\psi}(x)[D_N\psi](x) \quad \text{– No fermion doublers!} \]

- Invariance under infinitesimal chiral transformations:
  \[ \psi \rightarrow \psi + \epsilon \delta \psi, \quad \delta \psi = \gamma_5 (1 - \frac{1}{2}aD)\psi \]
  \[ \bar{\psi} \rightarrow \bar{\psi} + \delta \bar{\psi}\epsilon, \quad \delta \bar{\psi} = \bar{\psi}(1 - \frac{1}{2}aD)\gamma_5 \]

- \( D_N \) satisfies the Atiyah-Singer index theorem:
  \([\text{Hasenfratz, Laliena & Niedermayer 1998}]\]
  \[ \text{index}(D_N) = a^5 \sum_x \frac{1}{2} \text{Tr}(\gamma_5 D_N) = n_- - n_+ \]

\( D_N \) exhibits \(|n_- - n_+|\) exact zero modes

- But: numerical implementation of \( D_N \) expensive
Spectrum of the massless Neuberger-Dirac operator $D_N$

$$D_N = \frac{1}{a} (1 - U)$$

$$U = A/\sqrt{A^\dagger A} : \text{unitary}$$

→ eigenvalues lie on a circle
• Why bother about chiral symmetry?
  – allows clean study of spontaneous chiral symmetry breaking:

\[ \langle \bar{\psi}\psi \rangle \neq 0 \]

– allows direct comparison of lattice results with Chiral Perturbation Theory
– simplifies renormalisation patterns of local operators
2. Path integral and observables

Lattice formulation . . .

. . . preserves gauge invariance

. . . defines observables without reference to perturbation theory

. . . allows for stochastic evaluation of observables

Expectation value:

\[ \langle \Omega \rangle = \frac{1}{Z} \int D[U] D[\bar{\psi}] D[\psi] \Omega e^{-S_G[U] - S_F[U, \bar{\psi}, \psi]} \]

\[ = \frac{1}{Z} \int D[U] \Omega \prod_f \text{det} (\gamma_\mu D_\mu + m_f) e^{-S_G[U]} \]

\[ = \frac{1}{Z} \int \prod x, \mu dU_\mu(x) \Omega \prod_f \text{det} (D_{\text{lat}} + m_f) e^{-S_G[U]} \]
Monte Carlo simulation

1. Generate set of $N_c$ configurations of gauge fields $\{U_\mu(x)\}$, $i = 1, \ldots, N_c$, with probability distribution

$$W = \prod_f \det(D_{\text{lat}} + m_f) \ e^{-S_G[U]}$$

→ “Importance sampling”

→ Define an algorithm based on a Markov process:

Generate sequence $\{U\}_1 \rightarrow \{U\}_2 \rightarrow \ldots \rightarrow \{U\}_{N_c}$

Transition probability given by $W$

2. Evaluate observable for configuration $i$

$$\bar{\Omega} = \frac{1}{N_c} \sum_{i=1}^{N_c} \Omega_i, \quad \langle \Omega \rangle = \lim_{N_c \to \infty} \bar{\Omega}, \quad \text{statistical error: } \propto 1/\sqrt{N_c}$$
Dynamical quark effects

- $\det(D_{\text{lat}} + m_f)$: incorporates contributions of quark loops to $\langle \Omega \rangle$; non-local object; expensive to compute

- "Quenched Approximation:" $\det(D_{\text{lat}} + m_f) = 1$

→ Quark loops are entirely suppressed

Cost saving: \[ \frac{\text{"full" QCD}}{\text{quenched QCD}} = 100 - 1000 \]
Correlation functions

- Particle spectrum defined implicitly by correlation functions

- Consider $K$-Meson: $\phi_K(x) = s(x)\gamma_5\bar{u}(x)$

$$\sum_{\vec{x}} \langle \phi_K(x)\phi_K^\dagger(0) \rangle \sim 0 \left| \langle 0|\phi_K|K \rangle \right|^2 \left\{ e^{-m_Kx_0} + e^{-m_K[T-x_0]} \right\}$$

- Study asymptotic behaviour (large $x_0$) of Euclidean correlation functions:
  - exponential fall-off determines $m_K$
  - overall factor yields hadronic matrix element:
    $$\langle 0|\phi_K|K \rangle \propto F_K \quad \text{(kaon decay constant)}$$
Continuum limit

\[ \langle \Omega \rangle = \langle \Omega \rangle^{\text{lat}} + O(a^p), \quad p \in \mathbb{N}, \quad \text{lattice artefacts} \]

- Classical continuum limit: \( a \to 0 \)
- QFT: adjust the bare parameters as cutoff is removed, whilst keeping “constant physics”
  \[ a\Lambda = (b_0g_0^2)^{-b_1/2b_0^2} e^{-1/2b_0g_0^2} \ldots \]
- Perform simulations at several values of \( \beta \) and extrapolate to \( a = 0 \).

[Garden, Heitger, Sommer, H.W. 1999]

*fermion discretisation*

<table>
<thead>
<tr>
<th>Method</th>
<th>Order</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wilson</td>
<td>( O(a) )</td>
</tr>
<tr>
<td>Improved Wilson</td>
<td>( O(\alpha_s a), O(a^2) )</td>
</tr>
<tr>
<td>Staggered</td>
<td>( O(a^2) )</td>
</tr>
<tr>
<td>DWF, Neuberger</td>
<td>( O(a^2) )</td>
</tr>
</tbody>
</table>
3. Algorithms & Machines

- Challenge: design an efficient algorithms for dynamical quarks

- Workhorse: “Hybrid Monte Carlo”  
  \[ \text{[Duane, Kennedy, Pendleton, Roweth 1987]} \]

- Problem: algorithms become inefficient near chiral limit
  
  HMC works well down to  \( m_q \approx m_s/2 \Rightarrow m_\pi \approx 490 \text{ MeV} \)

- Solution of linear system:  
  \[ Ax = b \]

  becomes more demanding as condition number \( \lambda_{\text{max}}/\lambda_{\text{min}} \) increases

- Large autocorrelations: need larger number of update steps within Markov chain:

  \[ \{U\}_1 \rightarrow \ldots \{U\}_N, \quad \{U\}_1, \{U\}_N: \text{stat. independent}, \quad N \uparrow \text{ as } m_q \downarrow \]
• **Berlin 2001**: panel discussion on cost of dynamical fermion simulations

• Estimate cost to generate 1000 independent configurations (**Wilson quarks**)

![Graph showing the relationship between m/2 and a to calculate config costs](image)

- Simulations with Wilson quarks not practical for \( m_{\text{sea}} < m_s/2 \) and \( a < 0.1 \text{ fm} \)

→ Impossible to reach domain of “realistic” pion masses?
Recent progress

Alternative discretisations:

- **Improved staggered quarks**  
  [Lepage 1999; MILC, HPQCD 1999 –]

- **Domain Wall Fermions**  
  [Furman & Shamir 1993; RBC 1998 –]

- **twisted mass QCD**  
  [Frezzotti, Grassi, Sint, Weisz 2001; Frezzotti & Rossi 2003; DESY-Münster 2004 –]

Algorithmic improvements:

- **Schwarz alternating procedure**  
  [Lüscher 2003 – 05]

- **Mass precoditioning + multiple integration timescales**  
  [Hasenbusch, Jansen 2001; Peardon & Sexton 2002; Urbach et al. 2005]

- **Rational Hybrid Monte Carlo**  
  [Clark & Kennedy 2006]
Improved staggered quarks

- numerically more efficient than Wilson fermions

but: 4-fold degeneracy of fermion spectrum: 4 “tastes” per flavour
taste symmetry violations at $O(a^2)$ can be reduced → “improved”

→ Take fractional powers of fermion determinant: “fourth root trick”

$$2 + 1 \text{ light flavours: } \left\{ \det(D_{stag} + m_{u,d}) \right\}^{1/2} \times \left\{ \det(D_{stag} + m_s) \right\}^{1/4} e^{-S_G}$$

- Does this correspond to a local field theory?

- Does this lead to the correct continuum limit?

→ under debate . . .

[Davies et al., hep-lat/0304004]
Schwarz alternating procedure

Hermann Schwarz 1870

Solution of Dirichlet problem in complicated domains
Solve Laplace equation alternately in overlapping sub-domains

→ Domain Decomposition

Domain decomposition methods for QCD

- Dirichlet boundary conditions:
  block size $l \sim$ IR cutoff
  \[ q \geq \pi/l > 1 \text{GeV} \]

→ easy to simulate at all quark masses
Schwarz procedure for QCD achieves . . .

- “Natural” separation of local and global modes

- Algorithm scales slowly with quark mass

- Parallel efficiency:
  - blocks are mapped onto nodes of parallel computer
  - most CPU time spent on sub-domain
  → reduced communication overhead

Mode separation and improved scaling behaviour also achieved using

• Simulations with pion masses $\lesssim 300\,\text{MeV}$ are feasible on current parallel computers

• Wilson quarks no more difficult to simulate than staggered quarks

The Berlin Wall in 2006:
Machines

- Computing requirements: \( > 1 \text{ TFlops/s} \)
- Can be reached on *massively parallel systems*
- Lattice Dirac operators typically couples *nearest neighbours* → simple parallelisation
Computing platforms in Lattice QCD

• Commercial supercomputers:
  BlueGene/L, SGI Altix, IBM-p690, Hitachi SR8000, NEC Sx6, Fujitsu VPP700, . . .

• Custom made machines:
  CP-PACS $\sim 1\text{ TFlop/s}$ 1996 Tsukuba/Hitachi
  QCDOC $\sim 10\text{ TFlop/s}$ 2004 CU/UKQCD/Riken/IBM
  apeNEXT $\sim 10\text{ TFlop/s}$ 2005 INFN/DESY/Paris-Sud

• PC clusters + fast network:
  – Mass-produced components $\rightarrow$ cheap
  – Standard software + programming environment
Custom made machines I: apeNEXT

- Developed by INFN/DESY/Paris Sud

- Custom-designed processor
  - 8 Flops per cycle
  - 160 MHz $\Rightarrow$ 1.3 GFlops/s (peak)

- $\approx 40 - 50\%$ efficiency for QCD code

- Installations:
  - 1 rack = 512 nodes = 0.66 TFlops/s (peak)
    - INFN 12 racks
    - Bielefeld 6 racks
    - DESY 3 racks
    - Orsay 1 rack

- 0.6 €/MFlops/s (peak)
Custom made machines II: QCDOC

- Developed by Columbia/UKQCD/Riken/IBM
- IBM PowerPC 440 core + 64-bit FPU
  2 Flops per cycle, 400 MHz $\Rightarrow$ 0.8 GFlops/s (peak)
- $\approx 40 - 50\%$ efficiency for QCD code (assembly code generator)
- Installations:
  1 rack = 1024 nodes = 0.82 TFlops/s (peak)
  - Edinburgh: 14 racks
  - DOE: 14 racks
  - Riken/BNL: 13 racks
  - Columbia: 2.4 racks
  - Regensburg: 0.5 racks
- 0.5 $/MFlops/s$ (peak)
PC clusters

- **CPU:** Intel P4 XEON, AMD Opteron, DualCore
- **Node:** 1 – 2 CPUs, Rambus or DDR memory, local disks, . . .
- **Network:** Myrinet2000 (4 – 8 Gbit/s), Infiniband (10 – 20 Gbit/s) + switch
  \[
  \text{GigE (2 Gbit/s) + “mesh”}
  \]

- Typically larger latencies, smaller bandwidths
  \[\Rightarrow\] scalability not as good as for custom made machines
Code optimisation

- Cache management, memory prefetch
- Vector registers on Intel & AMD processors:

  **SSE, SSE2, (SSE3) “Streaming SIMD Extension”**

```c
for (i=0; i<100; i++)
a[i] = b[i] + c[i];

for (i=0; i<100; i+=4)
  __asm__ __volatile__ (
    "movaps %1, %%xmm0 \n\t"
    "movaps %2, %%xmm1 \n\t"
    "addps %%xmm0, %%xmm1 \n\t"
    "movaps %%xmm1, %0"
    : "=m" (a[i])
    : "m" (b[i]),
    "m" (c[i]));
```

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<table>
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<th>-0.6</th>
<th>2.9</th>
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</table>
Node performance and scaling

- Node performance as a function of the number of active processor cores:
### Large installations (Lattice QCD only)

<table>
<thead>
<tr>
<th>Location</th>
<th>Procs.</th>
<th>Network</th>
<th>[TFlops] (peak)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Wuppertal</td>
<td>1024 Opteron</td>
<td>GigE (2d)</td>
<td>3.7</td>
</tr>
<tr>
<td>JLab</td>
<td>384 Xeon</td>
<td>GigE (5d)</td>
<td>2.2</td>
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<tr>
<td>JLab</td>
<td>256 Xeon</td>
<td>GigE (3d)</td>
<td>1.4</td>
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<tr>
<td>FNAL</td>
<td>520 P4</td>
<td>Infiniband</td>
<td>3.4</td>
</tr>
<tr>
<td>FNAL</td>
<td>256 Xeon</td>
<td>Myrinet</td>
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</tr>
</tbody>
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Summary - part I

• Lattice formulation: “ab initio” method for QCD at low energies

• Conceptually well founded;
  truly non-perturbative procedure which respects gauge invariance

• Recent progress:
  – chiral symmetry + lattice regularisation
  – faster fermion algorithms
  – faster machines
  – refined methods (see later)
Lattice QCD — Recent Results
4. Hadron spectrum

Spectrum calculation:

- Choose bare parameters: coupling $\beta = 6/g_0^2$ and sea quark mass(es) $m_{\text{sea}}$

- Generate ensemble of gauge configurations

- Compute correlation functions:
  $$\sum_{\vec{x}} \langle \phi_{\text{had}}(\vec{x}) \phi_{\text{had}}^\dagger(0) \rangle \sim e^{-m_{\text{had}}x_0}$$

- $\phi_{\text{had}}(\vec{x})$: interpolating operator for given hadron:
  
  \begin{align*}
  K\text{-meson} & : \quad \phi_K = s \gamma_5 \bar{u}, \quad s \gamma_0 \gamma_5 \bar{u} \\
  \text{nucleon} & : \quad \phi_N = \varepsilon_{abc} \left( u^a C \gamma_5 d^b \right) u^c \\
  \Delta & : \quad \phi_\Delta = \varepsilon_{abc} \left( u^a C \gamma_\mu d^b \right) u^c
  \end{align*}
• Interpolating operators project on all states with the same quantum numbers:

\[
\sum_{\vec{x}} e^{i\vec{p} \cdot \vec{x}} \langle \phi_{\text{had}}(x) \phi_{\text{had}}^\dagger(0) \rangle = \sum_n w_n(\vec{p}) e^{-E_n(\vec{p})x_0}
\]

\[
w_n(\vec{p}) \equiv \frac{|\langle 0|\phi_{\text{had}}|n \rangle|^2}{2E_n(\vec{p})} : \text{spectral weight of } n^{\text{th}} \text{ state}
\]

• Excited states die out exponentially

• \(w_n(\vec{p})\) depends on particular choice of \(\phi_{\text{had}}\)
Eliminating the bare parameters

- Bare parameters: \( g_0, m_u, m_d, m_s, \ldots \)

- Can freely choose bare quark mass \( m_q \) in simulations;
  Which value of \( m_q \) corresponds to \( m_u, m_d, \ldots \)?

- Obtain hadron masses as functions of \( m_q \), e.g. \( a m_{PS}(m_{q_1}, m_{q_2}) \)

- Quark mass dependence of hadron masses:
  \[
  m_{PS}^2 \propto m_q, \quad m_V, m_N \propto m_q
  \]

- Eliminate the bare parameters in favour of hadronic input quantities:
  \[
  g_0 \sim 1/\ln a : \quad a^{-1} [\text{GeV}] = \frac{Q [\text{GeV}]}{(aQ)}, \quad Q = f_\pi, m_N, \Delta^r_{1P-1S}, \ldots
  \]
  \[
  \hat{m} = \frac{1}{2}(m_u + m_d) : \quad \frac{m_{PS}^2}{f_\pi^2} \rightarrow \frac{m_\pi^2}{f_\pi^2}, \quad m_s : \quad \frac{m_{PS}^2}{f_\pi^2} \rightarrow \frac{m_K^2}{f_\pi^2}
  \]
Hadronic renormalisation scheme

- Hadronic input quantities fix the values of the bare coupling and quark masses

Example:

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Quantity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$g_0$</td>
<td>$f_π$</td>
</tr>
<tr>
<td>$\frac{1}{2}(m_u + m_d)$</td>
<td>$m_π$</td>
</tr>
<tr>
<td>$m_s$</td>
<td>$m_K$</td>
</tr>
<tr>
<td>$m_c$</td>
<td>$m_{Ds}$</td>
</tr>
<tr>
<td>$m_b$</td>
<td>$m_{Bs}$</td>
</tr>
</tbody>
</table>

- Except for input quantities, all other observables are predictions
Light hadron spectrum in quenched QCD:


- Masses of lowest pseudoscalar & vector mesons; octet & decuplet baryons
- Wilson fermions; 4 values of $a$ \(\Rightarrow\) continuum extrapolation
- Lattice scale set by $m_\rho$

- Set strange quark mass by
  \[
  \frac{m_{2PS}^2}{m_\rho^2} = \frac{m_K^2}{m_\rho^2} \quad \text{("K-input")}
  \quad \text{or}
  \quad \frac{m_V^2}{m_\rho^2} = \frac{m_\phi^2}{m_\rho^2} \quad \text{("\phi-input")}
  \]

- Experimentally observed spectrum reproduced at the level of $10 - 15\%$

- Small but significant deviations
Dynamical quark effects:


- Can dynamical quark effects account for deviations?
- \( N_f = 2 + 1 \) flavours of improved staggered quarks; 2 values of \( a \)
- Lattice scale set by \( 1S - 1P \) splitting in \( \Upsilon \)

- Better consistency with experiment; More detailed study of baryons needed
- Complicated quark mass dependence for staggered quarks
- Detailed study of \textit{continuum limit} required
Glueballs

- **Glueballs**: hadron valence degrees of freedom are purely gluonic

- **Candidates**: $f_0(1370)$, $f_0(1500)$, $f_0(1710)$, $f_0(2220)$

- Are glueballs predicted by QCD? What are their **masses** and **widths**?

→ **Guidance from lattice calculations**

- Construct suitable interpolating operators; use **link variables** only

- Rotational symmetry broken on the lattice:
  - glueball operators constructed from representations of **hypercubic** group

→ ambiguous $J^{PC}$ assignment
**Glueball interpolating operators**

- contamination of higher spin states; e.g. $0^{++}$ can mix with $4^{++}$
- Matrix correlators:

  \[ C_{ij}(x_0) = \sum \langle O_i(x)O_j(0) \rangle \]

- \( \{O_1, \ldots, O_n\} \):
  basis of interpolating operators for given irrep. of the hypercubic group

→ Recover a given spin-parity in the continuum limit
• Projection properties of operators $O_i$ onto given state may differ

• Diagonalisation of the matrix correlator:

\[
\begin{pmatrix}
C_{11} & \cdots & C_{1n} \\
\vdots & \ddots & \vdots \\
C_{n1} & \cdots & C_{nn}
\end{pmatrix}(x_0) \rightarrow \begin{pmatrix}
C_1 \\
\vdots \\
C_n
\end{pmatrix}(x_0)
\]

\[
C_i(x_0) = \sum_{\vec{x}} \langle \Omega_i(\vec{x})\Omega_i(0) \rangle, \quad \Omega_i(\vec{x}) = \sum_{k=1}^{n} a_{ik} O_k(\vec{x})
\]

• Diagonalisation yields $C_i(x_0)$, i.e. appropriate linear combination of operators for the $i^{th}$ state in the spectral decomposition

→ More reliable ground state – necessary for higher excitations

• **Mixing** with scalar mesons – difficult to disentangle
Glueball spectrum in quenched QCD

\[ m_{0^{++}} = 1710(50)(80) \text{ MeV}, \quad m_{2^{++}} = 2390(30)(120) \text{ MeV} \]

\[ m_{0^{--}} = 2560(35)(120) \text{ MeV} \]

- No effects due to dynamical quarks or glueball-meson mixing
Pentaquarks:
The $\Theta^+$ (1540)

- Experimental situation unclear

- $\Theta^+(1540)$: narrow resonance(?) 100 MeV above $K - N$ threshold

- Do pentaquark states exist in QCD?
  - Is the lightest pentaquark consistent with $\Theta^+$?
  - What is its spin/parity?

- Challenge for lattice QCD: distinguish between a resonance and a 2-particle scattering state

- Interpolating operators for $\Theta^+ \sim uudd\bar{s}$ (spin-1/2):

  $$\chi_1^\mp = \varepsilon^{abc}(u^a C \gamma_5 d^b) [u^c(s^e \gamma_5 d^e) \mp \{u \leftrightarrow d\}]$$

  $$\chi_2 = \varepsilon^{gce} \varepsilon^{gfh} \varepsilon^{abc}(u^a C \gamma_5 d^b)(u^f C d^h) C^{-1}s^{Te}$$
• Spectral decomposition for bound state:

\[
\sum_{\vec{x}} \langle \chi(x) \chi^\dagger(0) \rangle = \sum_n w_n e^{-m_n x_0}, \quad w_n = \frac{|\langle 0 | \chi | n \rangle|^2}{2m_n}
\]

• Two non-interacting particles (zero total momentum):

\[
\sum_{\vec{x}} \langle \chi_1(x) \chi_2(x) \chi_1^\dagger(0) \chi_2^\dagger(0) \rangle = \sum_{n_1, n_2} \sum_{\vec{p}} \frac{1}{L^3} w_{n_1} w_{n_2} w_n(\vec{p}) e^{-(E_{n_1} + E_{n_2}) x_0}
\]

One-particle state: \( w_n \sim O(1), \ m_n = m_Q \)

Two-particle state: \( w_n(\vec{p}) \sim 1/L^3, \ E_n(\vec{p}) \equiv E_{KN}(\vec{p}) = E_K(\vec{p}) + E_N(\vec{p}) \)

\[
E_{KN}(\vec{p}) = \sqrt{m_K^2 + \vec{p}^2} + \sqrt{m_N^2 + \vec{p}^2}, \quad \vec{p} = \frac{n^2\pi}{L}
\]

→ Volume dependence reveals nature of the state
Several new results in 2005/06:
- quenched approximation;
- no continuum limit;
- study volume dependence

Matrix correlators crucial

Situation inconclusive:

[Csikor et al., hep-lat/0503012]

[Alexandrou+Tsapalis, hep-lat/0503013]
### spin-1/2 channel:

<table>
<thead>
<tr>
<th>Collab.</th>
<th>$a$ [fm]</th>
<th>matrix correlator</th>
<th>$m_{\pi}^{\text{min}}$</th>
<th>$L/a$</th>
<th>resonance exists</th>
<th>parity</th>
</tr>
</thead>
<tbody>
<tr>
<td>Csikor et al.</td>
<td>0.1</td>
<td>$14 \times 14$</td>
<td>400</td>
<td>20, 24</td>
<td>no</td>
<td>./</td>
</tr>
<tr>
<td>Alexandrou &amp; Tsapalis</td>
<td>0.1</td>
<td>$2 \times 2$</td>
<td>420</td>
<td>16, 24, 32</td>
<td>yes</td>
<td>−ve</td>
</tr>
<tr>
<td>Holland &amp; Juge</td>
<td>0.1</td>
<td>$2 \times 2$</td>
<td>300</td>
<td>18</td>
<td>no</td>
<td>./</td>
</tr>
<tr>
<td>Takahashi et al.</td>
<td>0.17</td>
<td>$2 \times 2$</td>
<td>500</td>
<td>8, 10, 12, 16</td>
<td>yes</td>
<td>−ve</td>
</tr>
<tr>
<td>Lasscock et al.</td>
<td>0.13</td>
<td>$2 \times 2$</td>
<td>460</td>
<td>20</td>
<td>no</td>
<td>./</td>
</tr>
<tr>
<td>Hagen et al.</td>
<td>0.15</td>
<td>$5 \times 5$</td>
<td>480</td>
<td>12</td>
<td>no</td>
<td>./</td>
</tr>
</tbody>
</table>

- Future studies:
  - smaller quark masses
  - large(r) matrix correlators
  - wider range of volumes

“Absence of evidence is not evidence of absence”  
[Holland + Juge, hep-lat/0504007]
5. Strong coupling constant

$\alpha_s$ – general procedure:

- Non-perturbative definition: $\alpha_X = \frac{g_X^2}{4\pi}$ \text{ scheme}
  
  e.g. $\alpha_{q\bar{q}}(1/r) = \frac{3}{4} r^2 \frac{dV(r)}{dr}$, \hspace{0.5cm} W(r, t) \approx \exp \{-V(r)t\}

- Evaluation & calibration: $\alpha_X(aq)$, \hspace{0.5cm} a.f. = #

- Running & matching: $\alpha_X^{(N_f)}(q) \to \alpha_{\overline{\text{MS}}}^{(N_f)}(q') \to \alpha_{\overline{\text{MS}}}^{(5)}(M_Z)$

- $\Lambda$-parameter: $\Lambda_X = \lim_{\mu \to \infty} \left\{ \mu(b_0g_X^2(\mu))^{-b_1/2b_0^2} \exp \left[-1/2b_0g_X^2(\mu)\right] \right\}$

- Systematic effects: \hspace{0.5cm} – Quality of matching $X \leftrightarrow \overline{\text{MS}}$
  \hspace{0.5cm} – control over scale evolution
  \hspace{0.5cm} – $N_f$-dependence
\( \alpha_s \) from heavy quarkonia

- **Definition:**
  \[ \alpha_V(q) = -\frac{3}{16\pi^2} q^2 V(q) \]  
  (heavy quark potential)

- **Perturbative expansion:**
  \[ -\ln \langle W_{nm} \rangle = c_{nm}^{(1)} \alpha_V(q^*) \left\{ 1 + c_{nm}^{(2)} \alpha_V(q^*) + c_{nm}^{(3)} \alpha_V(q^*)^2 + O(\alpha_V^3) \right\} \]

  \( c_{11}^{(1)}, c_{11}^{(2)} \): known in perturbation theory

  \( q^* \): “characteristic” momentum scale

\[ \langle W_{nm} \rangle \propto e^{-V(r)t} \]

\( r = na, \ t = ma \)

\( n \cdot m \): area of Wilson loop
• Conversion to $\overline{\text{MS}}$-scheme:

$$\alpha_{\overline{\text{MS}}}(e^{-5/6}q) = \alpha_V(q) + \frac{2}{\pi} \alpha_V(q)^2 + \ldots$$

• Calibration:

$$q [\text{GeV}] = \frac{(aq)}{(a\Delta^r_{1S-1P})} \Delta^r_{1S-1P} [\text{GeV}]$$

(radial $(1S - 2S)$ or orbital $(1S - 1P)$ splitting in $\Upsilon$-system)

• Matching & running: purely perturbative

Quark thresholds, e.g. for $N_f = 3$ dynamical flavours:

$$\alpha_{\overline{\text{MS}}}^{(3)}(7.5 \text{ GeV}) \rightarrow \alpha_{\overline{\text{MS}}}^{(3)}(m_c) = \alpha_{\overline{\text{MS}}}^{(4)}(m_c)$$

$$\rightarrow \alpha_{\overline{\text{MS}}}^{(4)}(m_b) = \alpha_{\overline{\text{MS}}}^{(5)}(m_b) \rightarrow \alpha_{\overline{\text{MS}}}^{(5)}(M_Z)$$
Recent results

- $N_f = 2 + 1$ flavours of improved staggered quarks

Compute $W_{11}, \ldots, W_{14}, \ldots, W_{23}$

Determine $\alpha_V^{(3)}$ from combined fit to data for $W_{nm}$ at 3 lattice spacings

$$\Rightarrow \quad \alpha_V^{(3)}(7.5 \text{ GeV}) = 0.2082(40)$$

Perturbative running, including quark thresholds, yields

$$\alpha_{\overline{\text{MS}}}^{(5)}(M_Z) = 0.1170 \pm 0.0012$$

- Lattice artefacts:

  Assume “window” where results are independent of $a = 0.17, 0.12, 0.09 \text{ fm}$
Comparison with other results: 

[PDG 2002: $\alpha_{\text{MS}}^{(5)}(M_Z) = 0.1172(20)$]

<table>
<thead>
<tr>
<th>$\alpha_{\text{MS}}^{(5)}(M_Z)$</th>
<th>$N_f$</th>
<th>Reference</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1170(12)</td>
<td>3</td>
<td>Mason et al., 2005</td>
</tr>
<tr>
<td>0.1076(20)(18)</td>
<td>0, 2</td>
<td>QCDSF/UKQCD, 2001</td>
</tr>
<tr>
<td>0.1118(17)</td>
<td>0, 2</td>
<td>SESAM, 1999</td>
</tr>
<tr>
<td>0.111(6)</td>
<td>0, 2</td>
<td>Wingate et al., 1997</td>
</tr>
<tr>
<td>0.1174(24)</td>
<td>0, 2</td>
<td>Davies et al., 1997</td>
</tr>
<tr>
<td>0.111(5)</td>
<td>0, 2</td>
<td>Aoki et al., 1995</td>
</tr>
<tr>
<td>0.105(4)</td>
<td>0</td>
<td>FNAL 1992</td>
</tr>
</tbody>
</table>
\( \alpha_s \) from the Schrödinger functional

[Lüscher, Narayanan, Weisz, Wolff 1992]

- Definition: 
  \[
  \frac{1}{g_{\text{SF}}^2} = \frac{1}{\Gamma_0[B]} \cdot \frac{\partial}{\partial \eta} \Gamma[B], \quad \Gamma[B] = -\ln \mathcal{Z}[C, C']
  \]

  \[
  \mathcal{Z}[C, C'] = \int D[A] D[\overline{\psi}] D[\psi] e^{-S[A, \overline{\psi}, \psi]}
  \]

  \rightarrow \text{yields } g_{\text{SF}} = g_0 \text{ for } \Gamma[B] = \Gamma_0[B]

- Schrödinger functional: finite-volume renormalisation scheme
  \( \overline{g}_{\text{SF}}^2 \) runs with the box size: \( \overline{g}_{\text{SF}}^2(L) \)

- Step scaling function: 
  \[
  \overline{g}_{\text{SF}}^2(sL) = \sigma(s, \overline{g}_{\text{SF}}^2(L))
  \]

  describes change in \( \overline{g}_{\text{SF}}^2 \) when the scale is changed by a factor \( s \).
• $\sigma$ can be computed in the continuum limit using recursive finite-size scaling

$$u_0 \equiv \bar{g}^2_{SF}(L_0), \quad u_{k+1} = \sigma(2, u_k)$$

• $L_{\text{max}}$: scale at which matching to infinite volume is performed:

$$L_{\text{max}} f_{\pi} = \# \quad \text{(calibration)}$$
• At the smallest coupling the evolution is continued perturbatively: \((\mu = 1/L)\)

\[
\Lambda_{\text{SF}} = \mu \left( b_0 \overline{g}^2_{\text{SF}} \right)^{-b_1/2b_0^2} e^{-1/2b_0 \overline{g}^2_{\text{SF}}} \times \exp \left\{ - \int_0^{\overline{g}_{\text{SF}}} dg \left[ \frac{1}{\beta(g)} + \frac{1}{b_0 g^3} - \frac{b_1}{b_0^2 g} \right] \right\}
\]

\[
\Lambda_{\text{MS}} = \Lambda_{\text{SF}} e^{c_1/(2b_0)}, \quad \overline{g}^2_{\text{MS}} = \overline{g}^2_{\text{SF}} \left( 1 + c_1 \overline{g}^2_{\text{SF}} + \ldots \right)
\]
Recursion + matching yields value for $\Lambda_{\text{MS}} \cdot L_{\text{max}}$

→ determine $L_{\text{max}}$ in physical units, e.g. $L_{\text{max}} f_\pi$, $L_{\text{max}}/r_0$, $r_0 = 0.5$ fm

<table>
<thead>
<tr>
<th>$N_f$</th>
<th>$\Lambda_{\text{MS}}^{(N_f)} r_0$</th>
<th>$\Lambda_{\text{MS}}^{(N_f)}$ [MeV]</th>
<th>Comments</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0.60(5)</td>
<td>238(19)</td>
<td>ALPHA 1999</td>
</tr>
<tr>
<td>2</td>
<td>0.62(4)(4)</td>
<td>245(16)(16)</td>
<td>ALPHA 2004</td>
</tr>
<tr>
<td>4</td>
<td>0.57(8)</td>
<td>225(32)</td>
<td>DIS NNLO &amp; $r_0 = 0.5$ fm</td>
</tr>
<tr>
<td>4</td>
<td>0.74(10)</td>
<td>292(39)</td>
<td>W.Av. &amp; $r_0 = 0.5$ fm</td>
</tr>
<tr>
<td>5</td>
<td>0.54(8)</td>
<td>213(32)</td>
<td>W.Av. &amp; $r_0 = 0.5$ fm</td>
</tr>
</tbody>
</table>

Running of $\alpha_s$ mapped out non-perturbatively, in the continuum limit

Large overlapping region between perturbative and non-perturbative regimes;
Perturbation theory applied only for scales $\mu \gtrsim 100$ GeV

Future: Calibration: improved value for $L_{\text{max}} f_\pi$; study $N_f = 3, \ldots$
6. Flavour physics

\[ V_{\text{CKM}} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \approx \begin{pmatrix} 1 - \frac{1}{2} \lambda^2 & \lambda & A \lambda^3 (\bar{\rho} - i \bar{\eta}) \\ -\lambda & 1 - \frac{1}{2} \lambda^2 & A \lambda^2 \\ A \lambda^3 (1 - \bar{\rho} - i \bar{\eta}) & -A \lambda^2 & 1 \end{pmatrix} \]

- Standard Model: 
  \( V_{\text{CKM}} \) is unitary \( 3 \times 3 \) matrix:
  \[ V_{ud} V_{ub}^* + V_{cd} V_{cb}^* + V_{td} V_{tb}^* = 0, \]

- Deviation from unitarity:
  signature for new physics

- Use experimental and theoretical input to overconstrain unitarity relations
• Relation between $\Delta M_d$, $\Delta M_s$, $\epsilon_K$ and CKM matrix elements:

$$\Delta M_d \propto f_{B_d}^2 \hat{B}_{B_d} |V_{td}V_{tb}^*|^2$$

$$\frac{\Delta M_s}{\Delta M_d} = \xi^2 \frac{m_{B_s}}{m_{B_d}} \frac{|V_{ts}|^2}{|V_{td}|^2}, \quad \xi = \frac{f_{B_s} \sqrt{\hat{B}_{B_s}}}{f_{B_d} \sqrt{\hat{B}_{B_d}}}$$

$$\epsilon_K \propto \hat{B}_K \text{Im}(V_{td}V_{ts}^*)$$

• $f_{B_d}, f_{B_s}$ hard to measure experimentally

• Theoretical determination of $\hat{B}_K, f_{Bq}, B_{Bq}$, $q = d, s$ afflicted with “hadronic uncertainties”

→ Lattice calculation treatment
\( K^0 - \overline{K}^0 \) mixing and \( \hat{B}_K \)

- Effective treatment of \( \Delta S = 2 \) transitions in QCD:

\[
\begin{align*}
\text{OPE} & \quad C(\mu) \, O^{\Delta S=2}(\mu) \\
\end{align*}
\]

- \( \hat{B}_K \) parameterises non-perturbative contribution to indirect CP violation:

\[
B_K(\mu) = \frac{\langle \overline{K}^0 | O^{\Delta S=2}(\mu) | K^0 \rangle}{\frac{8}{3} F_K^2 m_K^2}
\]

\[
O^{\Delta S=2} = [\bar{s}\gamma_\mu (1 - \gamma_5) d] [\bar{s}\gamma_\mu (1 - \gamma_5) d] = O^{\Delta S=2}_{VV+AA} - O^{\Delta S=2}_{VA+AV}
\]
Lattice calculations

- Compute 3-point correlation functions:

\[ \phi_K(x) = (s\gamma_0\gamma_5\bar{d})(x) \]

\[ \sum_{\vec{x}_i, \vec{x}_f} \left\langle \phi_K(x_f)O^{\Delta S=2}(0)\phi_K^+(x_i) \right\rangle \]

\[ \sim e^{-m_K(T-t_i)} e^{-m_K t_f} \frac{\lvert \zeta \rvert^2}{4m_K^2} \left\langle \bar{K}^0 \left| O^{\Delta S=2}(0) \right| K^0 \right\rangle \]

\[ m_K, \zeta = \left\langle 0 \left| s\gamma_0\gamma_5\bar{d} \right| K^0 \right\rangle \text{ known from 2-point function} \]

- Compute suitable ratios of 3- and 2-point functions:

\[ \frac{\sum_{\vec{x}_i, \vec{x}_f} \left\langle \phi_K(x_f)O^{\Delta S=2}(0)\phi_K^+(x_i) \right\rangle}{\sum_{\vec{x}_i} \left\langle \phi_K(x_i)\phi_K^+(0) \right\rangle \sum_{\vec{x}_f} \left\langle \phi_K(x_f)\phi_K^+(0) \right\rangle} \propto B_K(\mu) \]
Renormalisation and mixing

- Explicit chiral symmetry breaking:

\[ O_{VV+AA}(\mu) \] mixes under renormalisation:

\[
O_{VV+AA}^R(\mu) = Z(g_0, a_\mu) \left\{ O_{VV+AA}^{\text{bare}} + \sum_{i=1}^{4} \Delta_i(g_0) O_{i}^{\text{bare}} \right\}
\]

Wilson fermions: explicit chiral symmetry breaking: \( \Delta_i \neq 0 \)

Staggered fermions: Remnant chiral symmetry: \( \Delta_i = 0 \)

Domain Wall/Overlap: chiral symmetry preserved; expensive to simulate

- \( Z(g_0, a_\mu) \) and \( \Delta_i(g_0) \) can be computed non-perturbatively
Twisted mass QCD

\[
\begin{pmatrix} u \\ d \end{pmatrix} \rightarrow e^{i\pi \gamma_5 \tau_3/4} \begin{pmatrix} u \\ d \end{pmatrix}
\]

- No mixing problem: \( \Delta_i = 0 \):

\[
\langle K^0 | O_{VA+AV}^{\text{bare}} | K^0 \rangle_{\text{tmQCD}} = i \langle K^0 | O_{VV+AA}^{\text{bare}} | K^0 \rangle_{\text{QCD}}
\]

\( O_{VA+AV}^{\text{bare}} \): renormalised multiplicatively

- Non-perturbative determination of the total renormalisation factor, linking

\[
\langle K^0 | O_{VA+AV}^{\text{bare}} | K^0 \rangle_{\text{tmQCD}} \leftrightarrow \hat{B}_K
\]

[Grassi, Frezzotti, Sint & Weisz; Frezzotti & Rossi]

[ALPHA Collaboration, hep-lat/0505002]
Current status

- Quenched QCD, continuum limit, non-perturbative renormalisation:
  \[ B^\text{MS, NDR}_{K}(2 \text{ GeV}) = 0.573 \pm 0.034 \]
  \[ \Rightarrow \hat{B}_K = 0.789 \pm 0.046 \]

- Dynamical quark effects: \( N_f = 2, 2 + 1 \):
  no systematic continuum extrapolation, perturbative renormalisation, large errors
  \[ \text{[UKQCD, hep-lat/0406013; HPQCD, hep-lat/0603023]} \]

- Purple band: used in CKM fit
  \[ \hat{B}_K = 0.79 \pm 0.04 \text{ (gauss)} \pm 0.08 \text{ (flat)} \]
  \[ \text{[UTfit Collab. (Bona et al.), hep-ph/0606167]} \]
$B^0 - \bar{B}^0$ and $B_s^0 - \bar{B}_s^0$ mixing

$b$-quark on the lattice: multi-scale problem

$am_b \ll 1, \; Lm_\pi \gg 1, \; L/a \lesssim 50$

Computational strategies:

(1) **Static approximation**  
   - leading term in $1/m_Q$-expansion; expect corrections of order $\Lambda_{\text{QCD}}/m_Q$

(2) **Non-relativistic (NR) QCD**  
   - relativistic cutoff: $\Lambda_{\text{UV}} \sim a^{-1} \leq m_Q$; continuum limit does not exist

(3) **Extra/Interpolations in $m_Q$**  
   - use $m_Q \approx m_c$ and extrapolate to $m_b$  
   - can use static approximation as limiting case
4) “Fermilab approach” \cite{El-Khadra:Kronfeld:Mackenzie:2003}
   - use propagating heavy quarks
   - re-scaling of heavy quark fields interpolates between relativistic and non-relativistic regimes

5) Finite-size scaling \cite{de Divitiis:etal:2003:372}
   - use unphysically small volumes, keeping $am_Q \ll 1$
   - determine “distortion” due to finite volume
Recent results

(1) $f_{B_s}$ in quenched QCD

• $B_s$-meson: no chiral extrapolation required

• Compute renormalised matrix elements of

$$A_{0}^{\text{stat}} = (\bar{s} \gamma_0 \gamma_5 b)^{\text{stat}}$$

and

$$A_0 = (\bar{s} \gamma_0 \gamma_5 b)$$

for $m_{PS} = 1.7 - 2.6$ GeV

• Extrapolate to $a = 0$ at fixed $m_{PS}$

• Interpolate in $1/m_{PS}$ to $m_{B_s}$

HQ scaling law:

$$f_{B_s} m_{B_s}^{1/2} = C \left( \frac{M_b}{\Lambda} \right) \langle 0 | A_{0}^{\text{stat}} | B_s \rangle + O(1/M)$$

$$\Rightarrow f_{B_s} = 206 \pm 10 \text{ MeV}$$

(all errors except quenching)
(2) Chiral logarithms in $f_{B_s}/f_{B_d}$, $\xi$

- SU(3)-flavour-breaking ratios can be computed in ChPT:

$$\frac{f_{B_s}}{f_{B_d}} - 1 = (m_K^2 - m_\pi^2) f_2(\mu) - \frac{1 + 3g^2}{4\pi f_0^2} \left[ \frac{1}{2} I_P(m_K) + \frac{1}{4} I_P(m_\eta) - \frac{3}{4} I_P(m_\pi) \right]$$

$$I_P(m_{PS}) = m_{PS}^2 \ln(m_{PS}^2/\mu^2), \quad g^2 = g_{B^*B\pi}^2$$

- No deviation from linear mass dependence if $m_{\text{light}} \gtrsim m_s/2$

- Inclusion of chiral logs can drastically change $f_{B_s}/f_{B_d}$

$$\frac{f_{B_s}}{f_{B_d}} = \begin{cases} 1.16 \pm 0.05 & \text{without chiral log,} \quad [\text{Ryan @ Lattice 01}] \\ 1.32 \pm 0.10 & \text{with chiral log,} \quad [\text{Kronfeld & Ryan, hep-ph/0211271}] \end{cases}$$

- Chiral logs in $f_{B_s}/f_{B_d}$ and $F_K/f_\pi$ are nearly of the same size

  [Bećirević et al., hep-ph/0211271]

  Expect $f_{B_s}/f_{B_d} \approx F_K/f_\pi = 1.22$
• **Improved staggered** quarks, \( N_f = 2 + 1 \):

  some evidence for chiral logs

  \[
  \frac{f_{B_s}}{f_{B_d}} = 1.20(3)(1)
  \]

  [Okamoto @ Lattice 05, hep-ph/0510113]

• Curvature due to combining two data sets with different systematics?

  → More precise data near chiral limit;
  different discretisations;
  continuum extrapolation
Summary - part II

• Lattice QCD: mature field of research:
  – provides guidance for experimental searches: spectroscopy, QCD phase diagram, . . .
  – provides input for phenomenology: fundamental parameters, flavour physics
  – understanding of hadron structure from first principles,
  – test of other theoretical methods: Chiral Perturbation Theory

A lot of work remains to be done!